CONVECTIVE HEAT TRANSFER IN THE NONSTATIONARY MOTION OF A MAXWELLIAN FLUID BETWEEN PARALLEL PLANES

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The convective heat-transfer problem is investigated for a Maxwellian fluid in generalized Couette flow in the case of large transient times for the process.

The problems of viscoelastic fluid flow have been handled rather successfully in recent years. The hydrodynamical aspects of the problem have been covered in a great many papers, of which we cite [1-4]. The authors of these papers have investigated the flow of both linear viscoelastic media (Maxwellian model) and more complex nonlinear viscoelastic fluids. However, the problems of convective heat transfer in the motion of viscoelastic fluids has been almost completely ignored; only in [5] has the heat-transfer problem been solved for a free-convective flow of a viscoelastic fluid. Viscoelastic properties are inherent in many polymer solutions and melts, lending considerable practical significance to the problems of convective heat transfer in the motion of viscoelastic fluids. In the present article we adopt the simple rheological model of Maxwell as our model of a viscoelastic fluid; the flow geometry is characterized by the conditions of the generalized Coutte problem.

We choose a coordinate system oxy so that the plane y = 0 is congruent with the lower plane, which is assumed to be at rest, the x axis is directed along the motion of the upper plane, and the y axis is perpendicular to the planes. We assume that at $t < 0$ the medium is at rest and its temperature is constant; at $t = 0$ the upper plane begins to move, and simultaneously a longitudinal pressure gradient and thermal field are applied (the wall temperature jumps abruptly to T_W and remains constant thereafter, forming boundary conditions of the first kind). We neglect axial heat propagation. This formulation corresponds to the following mathematical statement of the problem:

a) equation of motion and rheological model equation [6, 7]:

$$-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}=\rho\frac{\partial u}{\partial t};\qquad \theta\frac{\partial \tau}{\partial t}+\tau=\mu\frac{\partial u}{\partial y} \tag{1}$$

subject to the boundary conditions

$$u(0,\ y)=0;\quad u(t,\ 0)=0;\quad u(t,\ h)=U; \tag{2}$$

b) thermal-conduction equation:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}=a\frac{\partial^2 T}{\partial y^2} \tag{3}$$

subject to the initial and boundary conditions

$$T(0,\ x,\ y)=T_0;\quad T(t,\ 0,\ y)=T_0;\quad T(t,\ x,\ 0)=T_w;\quad T(t,\ x,\ h)=T_w. \tag{4}$$

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We confine the ensuing analysis to the case of constant ρ, μ, and θ. In this formulation the rheodynamical problem is autonomous and is solved independently of the thermodynamical problem.

A. Rheodynamics

The solution of the set of Eqs. (1) under the initial and boundary conditions (2) for the case of a constant longitudinal pressure gradient

$$\frac{dp}{dx} = P = \text{const} > 0$$

can be obtained by superposition of the solutions for simple Couette flow and flow in a plane tube, from [7]:

$$u(t, y) = U\left[\frac{y}{h} + \frac{2}{\pi}\exp\left(-\frac{t}{2\theta}\right)\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\left(\operatorname{ch} r_m t + \frac{\operatorname{sh} r_m t}{2\theta r_m}\right)\sin\frac{m\pi y}{h}\right] - \frac{P}{2\mu}\left\{y(h-y) - \frac{4h^2}{\pi^3}\exp\left(-\frac{t}{2\theta}\right) \times \sum_{m=1}^{\infty}\frac{1-(-1)^m}{m^3}\left[\operatorname{ch} r_m t + \operatorname{sh} r_m t\left(\theta r_m + \frac{1}{4\theta r_m}\right)\right]\sin\frac{m\pi y}{h}\right\}, \tag{5}$$

where

$$r_m = \frac{1}{2\theta}\sqrt{1 - \frac{4\pi^2 m^2 \mu\theta}{h^2\rho}}. \tag{6}$$

As $t \to \infty$ the velocity profile of the viscoelastic medium goes over to the steady-state velocity distribution of the generalized Couette problem for a viscous fluid:

$$u = u_\infty = U\frac{y}{h} - \frac{P}{2\mu}y(h-y). \tag{7}$$

We analyze the configuration of the velocity profile for large times. We introduce the parameter

$$K = \frac{2\pi^2\mu\theta}{h^2\rho} \tag{8}$$

to obtain, assuming $K \ll 1$,

$$r_1 = \frac{1}{2\theta}\left(1 - K + \frac{K^2}{2}\right). \tag{9}$$

Retaining only the first terms of the series, we have

$$u = u_\infty - A\exp\left(-b\frac{t}{\theta}\right)\sin\frac{\pi y}{h}, \tag{10}$$

where

$$A = \frac{2}{\pi}\left[\frac{2+K}{2}U - \frac{2h^2}{\pi^2}\frac{P}{\mu}\right]; \quad b = \frac{K(2-K)}{4}. \tag{11}$$

Let us estimate the order of magnitude of t_1, for which it is admissible to retain only the first term in the series. It follows from Eq. (5) that at a fixed time the velocity profile for Poiseuille flow [the second series in Eq. (5)] is more precisely approximated by retention of just the first term of the series than the profile for Couette flow [the first series in Eq. (5)]. Therefore, the time is estimated from the first series. The calculations indicate that for

$$t_1 \simeq \frac{2\theta}{K} = \frac{h^2\rho}{\pi^2\mu} \tag{12}$$

the first rejected term is an order of magnitude smaller than the first term of the first series, and the latter term, on the other hand, is considerably smaller than U(y/h) for large times. Consequently, for $t > t_1$, where t_1 is determined by condition (12), it is permissible to approximate the velocity profile (5) by the distribution (10).

B. Heat Transfer

We introduce the dimensionless variables and parameters

$$T' = \frac{T - T_w}{T_0 - T_w}; \quad t' = \frac{t}{\theta}; \quad y' = \frac{y}{h}; \quad u' = \frac{u}{U}; \quad x' = \frac{x}{U\theta}; \quad m^2 = \frac{ah^2}{\theta} \tag{13}$$

(we omit the primes from the ensuing calculations). The substitution of (13) into (3) and (4) and into (10) and (11), respectively, yields the equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = m^2 \frac{\partial^2 T}{\partial y^2} \tag{14}$$

subject to the initial and boundary conditions

$$T(0, x, y) = 1; \quad T(t, x, 0) = 0; \quad T(t, 0, y) = 1; \quad T(t, x, 1) = 0, \tag{15}$$

where

$$u = u(t, y) = u_\infty(y) - v(t, y); \tag{16a}$$

$$v \ll u_\infty; \tag{16b}$$

$$v(t, y) = A\exp(-bt)\sin \pi y; \tag{17}$$

$$u_\infty(y) = y - \frac{Ph^2}{2\mu U} y(1-y); \tag{18}$$

$$A = \frac{2}{\pi}\left[\frac{2+K}{2} - \frac{2h^2}{\pi^2}\frac{P}{U\mu}\right]. \tag{19}$$

The integration of Eq. (14) in the general case is difficult. We therefore seek an approximate solution of the problem for large times. We represent T(t, x, y) in the form

$$T(t, x, y) = T_\infty(x, y) - \varphi(t, x, y) \tag{20a}$$

$$(\text{for large times } T \to T_\infty; \quad \varphi \ll T_\infty). \tag{20b}$$

We find the stationary temperature distribution from the solution of the equation

$$u_\infty(y)\frac{\partial T_\infty}{\partial x} = m^2 \frac{\partial^2 T_\infty}{\partial y^2} \tag{21}$$

subject to the boundary conditions

$$T_\infty(0, 1) = 1; \quad T_\infty(x, 0) = 0; \quad T_\infty(x, 1) = 0. \tag{22}$$

Equation (21) with the boundary conditions (22) is solved by the separation of variables (Fourier method) [8]. In view of the difficulties associated with finding the exact eigenvalues and eigenvectors we use an approximative method, namely the so-called WKB (Wentzel–Kramers–Brillouin), or phase-integral, method used in [9-11]. The solution of problem (21) has the form

$$T_\infty(x, y) = \sum_{n=0}^{\infty} C_n \psi_n(y) \exp[-\varepsilon_n^2 m^2 x]. \tag{23}$$

The eigenfunctions are determined from the equation

$$\psi''(y) + \varepsilon^2 u_\infty(y)\psi(y) = 0 \tag{24}$$

subject to the boundary conditions

$$\psi(0) = 0; \tag{25a}$$

$$\psi(1) = 0. \tag{25b}$$

The condition for the entry temperature T(t, 0, y) = 1 is correctly stated only in the case of a low counter-pressure:

$$P < 2\mu \frac{U}{h^2}. \tag{26}$$

Regarding ε as a large parameter, we apply the first-order WKB approximation [12]:

$$\psi(y) = \frac{1}{\varepsilon^{1/2}\left[u_\infty(y)\right]^{1/4}} \left\{A_1 \exp\left[i\varepsilon \int_y^1 \sqrt{u_\infty(y)}\,dy\right] + B_1 \exp\left[-i\varepsilon \int_y^1 \sqrt{u_\infty(y)}\,dy\right]\right\}. \tag{27}$$

Under condition (26) $u_\infty(y) > 0$ everywhere in the channel, i.e., we have only one turning poing [13], y = 0, because u(0) = 0. Representing the solution in the form

$$\psi(y) = \frac{1}{\varepsilon^{1/2}\left[u_\infty(y)\right]^{1/4}} \left\{A \sin\left[\varepsilon \int_y^1 \sqrt{u_\infty(y)}\,dy\right] + B \cos\left[\varepsilon \int_y^1 \sqrt{u_\infty(y)}\,dy\right]\right\}, \tag{28}$$

we obtain the following upon satisfaction of (25b):

$$\psi(y) = \frac{\sin\left[\varepsilon \int_y^1 \sqrt{u_\infty(y)}\,dy\right]}{\varepsilon^{1/2}\left[u_\infty(y)\right]^{1/4}}. \tag{29}$$

Inasmuch as y = 0 is a turning point, in the vicinity of that point the WKB approximation is inapplicable. For small y we linearize the velocity profile, whereupon, taking the condition of nondetachment at the wall into account, we obtain

$$u_\infty(y) = u_\infty(0) + u'_\infty(0)\,y + \ldots = u'_\infty(0)\,y. \tag{30}$$

The substitution of (30) into (24) leads to the equation

$$\psi''(y) + \varepsilon^2 y u'_\infty(0)\,\psi(y) = 0, \tag{31}$$

whose solution can be represented in terms of Bessel functions:

$$\psi(y) = C\sqrt{y}\,J_{1/3}\left[\frac{2\varepsilon}{3}\sqrt{u'_\infty(0)}\,y^{3/2}\right] + D\sqrt{y}\,J_{-1/3}\left[\frac{2\varepsilon}{3}\sqrt{u'_\infty(0)}\,y^{3/2}\right]. \tag{32}$$

For large ε (such that $\varepsilon y^{3/2}$ is large for small y) we have the following asymptotic representation of the Bessel functions [14]:

$$\begin{aligned} J_{1/3}\left[\frac{2\varepsilon}{3}\sqrt{u'_\infty(0)}\,y^{3/2}\right] &= \sqrt{\frac{3}{\pi\varepsilon\sqrt{u'_\infty(0)}\,y^{3/2}}}\cos\left[\frac{2\varepsilon}{3}\sqrt{u'_\infty(0)}\,y^{3/2} - \frac{5\pi}{12}\right], \\ J_{-1/3}\left[\frac{2\varepsilon}{3}\sqrt{u'_\infty(0)}\,y^{3/2}\right] &= \sqrt{\frac{3}{\pi\varepsilon\sqrt{u'(0)}\,y^{3/2}}}\cos\left[\frac{2\varepsilon}{3}\sqrt{u'_\infty(0)} - \frac{\pi}{12}\right]. \end{aligned} \tag{33}$$

The insertion of (33) into (32) and "matching" of the WKB solution to the solution obtained near the turning point with regard for the boundary condition (25a) gives us the representations we seek for the eivenvalues:

$$\varepsilon_n = \left(n + \frac{11}{12}\right)\pi\left[\int_0^1 \sqrt{u_\infty(y)}\,dy\right]^{-1} \tag{34}$$

and the coefficients:

$$C = \frac{2\sqrt{\pi}}{3}\cos\left[\varepsilon\int_0^1 \sqrt{u_\infty(y)}\,dy - \frac{\pi}{12}\right], \tag{35}$$
$$D = 0.$$

As a result, the eigenfunctions in the interval $0 < y \leq 1/4$ are found from (32) with (34) and (35) taken into account, and in the interval $1/4 < y < 1$ from (29) and (34) (the choice of $y = 1/4$ as the crossover is conditional).

We now determine the nonstationary temperature "defect." Upon substitution of (16a) and (20a) into (14) with regard for the stationary-temperature equation (21) and omission of the term containing the product of small quantities $(v\partial\varphi/\partial x)$, we obtain the equation we are after:

$$\frac{\partial\varphi}{\partial t} + u_\infty(y)\frac{\partial\varphi}{\partial x} - m^2\frac{\partial^2\varphi}{\partial y^2} = -v(t, y)\frac{\partial\theta}{\partial x}. \tag{36}$$

We find the boundary conditions from (16a) and (22):

$$\varphi(t, 0, y) = 0; \quad \varphi(t, x, 0) = 0; \quad \varphi(t, x, 1) = 0. \tag{37}$$

Physical conditions indicate that it is necessary to impose a condition on the time for φ as $t \to \infty$, because for small values of t the assumptions (16b) and (20b), which were injected into the scheme of the solution, are inapplicable. It is clear that

$$\varphi|_{t\to\infty} = 0. \tag{38}$$

The right-hand side of Eq. (36) can be determined from (17) and (23). As a result,

$$\frac{\partial\varphi}{\partial t} + u_\infty(y)\frac{\partial\varphi}{\partial x} - m^2\frac{\partial^2\varphi}{\partial y^2} = -A\exp[-bt]\sin\pi y\sum_{n=0}^{\infty} C_n\varepsilon_n^2 m^2\psi_n(y)\exp[-\varepsilon_n^2 m^2 x]. \tag{39}$$

We seek the solution of (39) in the form

$$\varphi(t, x, y) = \exp[-bt]\,\theta_1(x, y). \tag{40}$$

Condition (38) is now satisfied automatically. For the function $\theta_1(x, y)$ we deduce the equation

$$m^2\frac{\partial^2\theta_1}{\partial y^2} - u_\infty(y)\frac{\partial\theta_1}{\partial x} + b\theta_1 = \sum_{k=0}^{\infty} f_k(y)\exp[-\varepsilon_k^2 m^2 x], \tag{41}$$

in which

$$f_k(y) = AC_k\sin\pi y\,\varepsilon_k^2 m^2\psi_k(y) \tag{42}$$

subject to the boundary conditions

$$\theta_1(0, y) = 0; \quad \theta_1(x, 0) = 0; \quad \theta_1(x, 1) = 0. \tag{43}$$

The solution of the inhomogeneous equation (41) under the homogeneous boundary conditions (43) is found as a Fourier series on the eigenfunctions of the corresponding homogeneous equation:

$$\theta_1(x, y) = \sum_{n=0}^{\infty} D_n(\lambda_n, x)\,\psi_{1n}(y)\,X_n(x). \tag{44}$$

The corresponding homogeneous equation

$$m^2\frac{\partial^2\theta_1}{\partial y^2} - u_\infty(y)\frac{\partial\theta_1}{\partial x} + b\theta_1 = 0 \tag{45}$$

is solved by the Fourier method. For the eigenfunction we obtain

$$\psi_1''(y) + \frac{1}{m^2}\left[b + \lambda^2 u_\infty(y)\right]\psi_1(y) = 0; \tag{46}$$

$$\psi_1(0) = 0; \quad \psi_1(1) = 0. \tag{47}$$

Since m is a small quantity, we seek the solution of (46) by the WKB method. We obtain as a result

$$\psi_1 = A\,\frac{\sin\left[\frac{1}{m}\int\limits_y^1 \sqrt{b+\lambda^2 u_\infty}\,dy\right]}{m^{-1/2}(b+\lambda^2 u_\infty)^{1/4}} + B\,\frac{\cos\left[\frac{1}{m}\int\limits_y^1 \sqrt{b+\lambda^2 u_\infty}\,dy\right]}{m^{-\frac{1}{2}}(b+\lambda^2 u_\infty)^{1/4}}. \tag{48}$$

Inasmuch as $u_\infty(y) > 0$ for $0 < y \le 1$, there is no turning point for the solution (48), because $b > 0$. Once (47) is satisfied, we have

$$B = 0$$

and

$$\sin\left[\frac{1}{m}\int\limits_0^1 \sqrt{b+\lambda^2 u_\infty(y)}\,dy\right] = 0, \tag{49}$$

whence

$$\int\limits_0^1 \sqrt{b+\lambda_n^2 u_\infty(y)}\,dy = n\pi m. \tag{50}$$

The solution of (50) for λ_n gives us the representation we seek for the eigenvalues. The eigenfunctions in the entire channel are found from the relation

$$\psi_{1n} = \frac{\sin\left[\frac{1}{m}\int\limits_y^1 \sqrt{b+\lambda_n^2 u_\infty(y)}\,dy\right]}{m^{-\frac{1}{2}}\left[b+\lambda_n^2 u_\infty(y)\right]^{1/4}}. \tag{51}$$

To find the coefficients D_n in Eq. (44) we expand the function

$$\varphi_k(y) = \frac{f_k(y)}{u_\infty(y)}, \tag{52}$$

where $f_k(y)$ is determined by Eq. (42), in a Fourier series on the complete system of orthogonal functions $\psi_{1n}(y)$:

$$\varphi_k(y) = \sum_{n=0}^{\infty} a_{nk}\psi_{1n}(y), \tag{53}$$

where

$$a_{nk} = \frac{\int\limits_0^1 \frac{f_k(y)}{u_\infty(y)}\left[b+\lambda_n^2 u_\infty(y)\right]\psi_{1n}(y)\,dy}{\int\limits_0^1 \psi_{1n}^2(y)\left[b+\lambda_n^2 u_\infty(y)\right]dy}. \tag{54}$$

The insertion of (53) and (44) into (41) with regard for the eigenfunction equation (46) yields the following equation for D_n after suitable algebraic transformations:

$$\frac{dD_n}{dx} = -\sum_{k=0}^{\infty} a_{nk}\exp(-\varepsilon_k^2 m^2 x + \lambda_n^2 x) \tag{55}$$

subject to the boundary conditions

$$D_n(0) = 0. \tag{56}$$

The integration of (55) and satisfaction of (56) yield

$$D_n = -\sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_n^2 - \varepsilon_k^2 m^2}\left\{\exp\left[(\lambda_n^2 - \varepsilon_k^2 m^2)x\right] - 1\right\}. \tag{57}$$

As a result, we obtain the desired temperature distribution:

$$T(t,\ x,\ y) = \sum_{n=0}^{\infty} C_n \psi_n(y) \exp(-\varepsilon_n^2 m^2 x)$$

$$-\exp(-bt) \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{nk} \exp[-\lambda_n^2 x]}{\lambda_n^2 - \varepsilon_k^2 m^2} \{\exp[(\lambda_n^2 - \varepsilon_k^2 m^2)\, x] - 1\}\, \psi_{1n}(y). \qquad (58)$$

This solution is also applicable to the case $P < 0$. In this case there are no constraints on the quantity $|P|$.

NOTATION

x, y	are the axes of the Cartesian coordinate system;
p	is the pressure;
τ	is the shear stress;
ρ	is the density of the medium;
π	are the dimensionless quantities;
θ	is the relaxation time;
u	is the projection of velocity on the x axis;
h	is the separation of the bounding planes;
U	is the velocity of the upper plane;
T_0	is the temperature of the medium at entry or initial temperature;
P	is the longitudinal pressure gradient;
T_W	is the wall temperature of the channel;
a	is the thermal diffusivity;
T_∞	is the stationary temperature distribution;
u_∞	is the stationary velocity profile;
φ	is the nonstationary temperature "defect";
v	is the nonstationary velocity "defect";
ε_n, ψ_n	are the eigenvalues and eigenfunctions of the Sturm–Liouville problem (stationary case);
$J_p(z)$	are the Bessel functions of the first kind;
λ_n	are the eigenvalues of the Sturm–Liouville problem (nonstationary case).

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